7 Inter-symbol Interference and Pulse Shaping

7.1. Recall that, in ***Pulse-Amplitude Modulation*** (PAM), we start with a sequence of numbers

discrete-time message

$$\cdots, m[-3], m[-2], m[-1], m[0], m[1], m[2], m[3], \cdots$$

as shown in Figure 36.

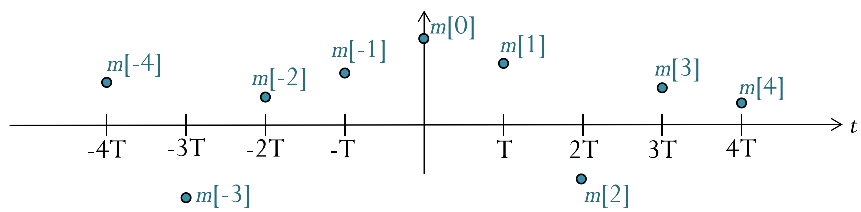


Figure 36: Sequence of Numbers for PAM

- These numbers may come from sampling a continuous-time signal $m(t)$. Alternatively, it may directly represent (digital) information that intrinsically available in discrete-time.
- Because the $m[n]$ may not come from sampling, we call each $m[n]$ a **symbol**.

We use these numbers to modify (modulate) the height (amplitude) of a pulse train. A single pulse is denoted by $p(t)$. This pulse occurs every T seconds.

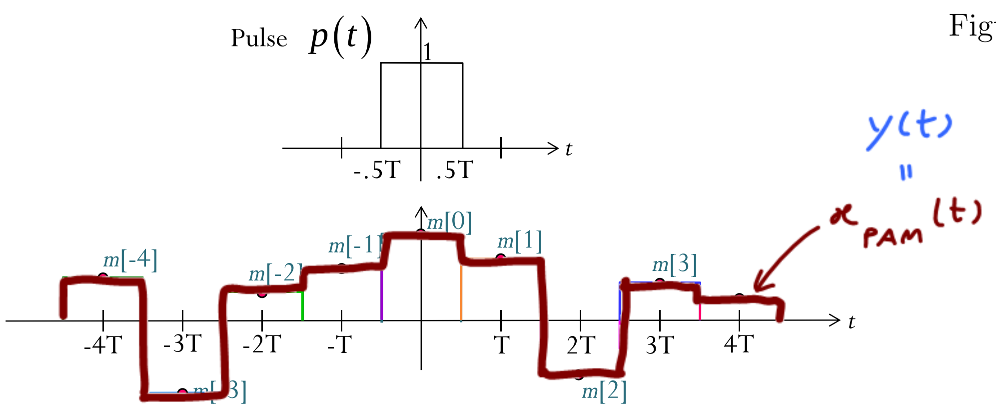
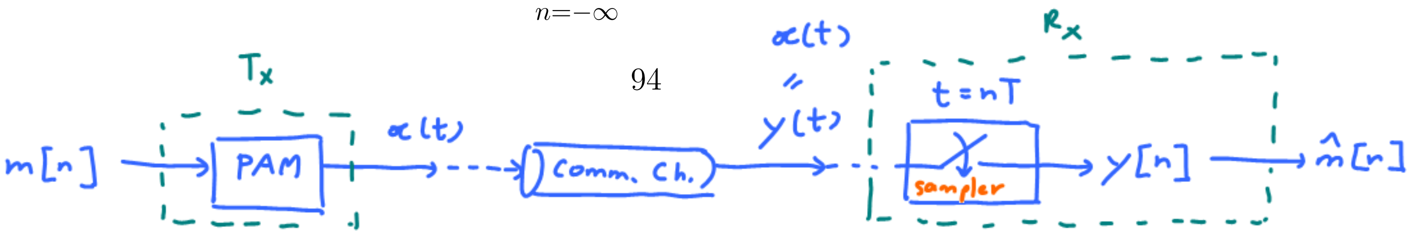


Figure 37: PAM

The PAM signal is then

$$x(t) = \sum_{n=-\infty}^{\infty} m[n]\, p(t - nT)$$



This signal is transmitted via the communication channel which usually corrupts it. At the receiver, the received signal is $y(t)$. A more advanced receiver would try to first cancel the effect of the channel. However, for simplicity, let's assume that our receiver simply samples $y(t)$ every T seconds to get

$$y[n] = y(t)|_{t=nT}$$

and we will take this to be the estimate $\tilde{m}[n]$ of our $m[n]$.

- If $m[n]$ is the sampled version of $m(t)$, then at the receiver, after we recover $m[n]$, we can reconstruct $m(t)$ by using the reconstructing equation (58).

Because our assumed receiver is so simple, we are going to also assume[25] that $y(t) = x(t)$.

7.2. In this section, our goal is to design a "good" pulse $p(t)$ that satisfies two important properties

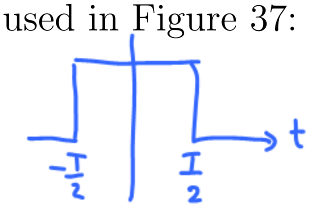
(a) $\tilde{m}[n] = m[n]$ for all n. Under our assumptions above, this means we want $x[n] \equiv x(nT) = m[n]$ for all n.

(b) $P(f)$ is band-limited and hence $X(f)$ is band-limited.

We will first give examples of "poor" $p(t)$.

Example 7.3. Let's consider the rectangular pulse used in Figure 37:

$$p(t) = 1[|t| \le T/2].$$



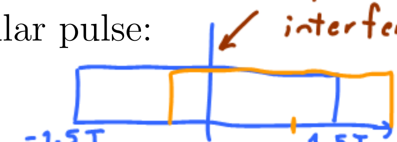
(a) $\tilde{m}[n] = m[n]$ for all n.

(b) The Fourier transform of the rectangular pulse is a sinc function. So, it is not band-limited.

Example 7.4 (Slide)**.** Let's try a wider rectangular pulse:

$$p(t) = 1[|t| \le 1.5T].$$

The pulse at T interferes with the pulse at 0.



Here, we face a problem called **inter-symbol interference (ISI)** in our sequence $\tilde{m}[n]$ at the receiver. The pulses are too wide; they interfere with other pulses at the sampling time instants (decision making instants), making $\tilde{m}[n] \ne m[n]$.

[25] Alternatively, we may assume that there is an earlier part of the receiver that (perfectly) eliminates the effect of the channel for us.

Example 7.5 (slide). $p(t) = 1[|t| \le T/4]$.

- When the pulse $p(t)$ is narrower than T, we know that the pulses in PAM signal will not overlap and therefore we won't have any ISI problem.

Example 7.6 (slides).

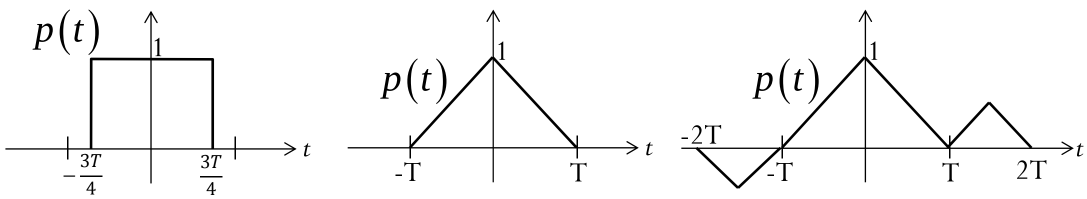
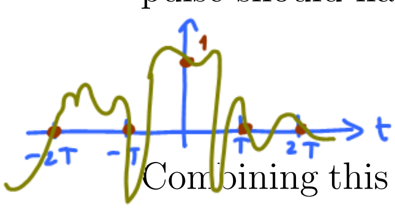


Figure 38: Examples of pulses that do not cause ISI.

- Even when the pulses are wider than T, if they do not interfere with other pulses at the sampling time instants (decision making instants), we can still have no ISI.

(criterion)
first form of the requirement ↑ in the time domain

7.7. We can now conclude that a ① "good" pulse satisfying condition (a) in 7.2 must not cause **inter-symbol interference (ISI)**: at the receiver, the nth symbol $\tilde{m}[n]$ should not be affected by the preceding or succeeding transmitted symbol $m[k]$, $k \neq n$. This requirement means that a "good" pulse should have the following property:

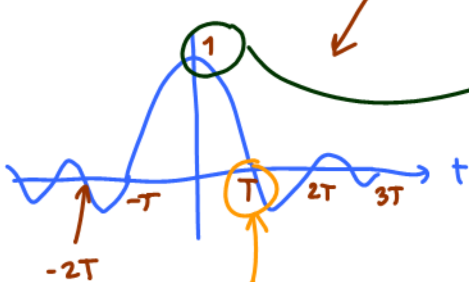


$$p[n] \equiv p(t)|_{t=nT} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{59}$$

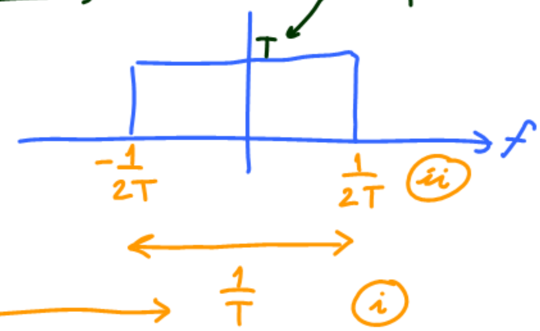
$\delta[n]$ ← discrete-time version of the Dirac delta func.

Combining this with condition (b) in 7.2, we then want ② "band-limited pulses specially shaped to avoid ISI (by satisfying (59))" [3, p 506].

7.8. An obvious choice for such $p(t)$ would be the sinc function that we used in the reconstruction equation (58): (iii)



area = 1 = h × $\frac{1}{T}$



Recall Figure 33, repeated here (with modified labels) as Figure 39.

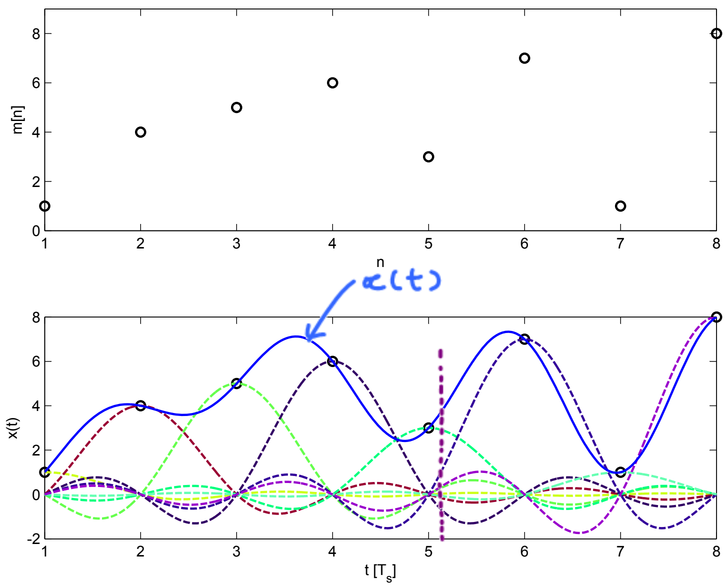


Figure 39: Using the sinc pulse in PAM

Practically, there are problems that force us to seek better pulse shape.

(a) Infinite duration (Does not decay fast enough)

(b) Steep slope at each 0-intercept. ⇒ sensitive to timing/sync. error

(c) $\max_t \{x(t)\}$ could be a lot larger than $\max_n \{m[n]\}$.

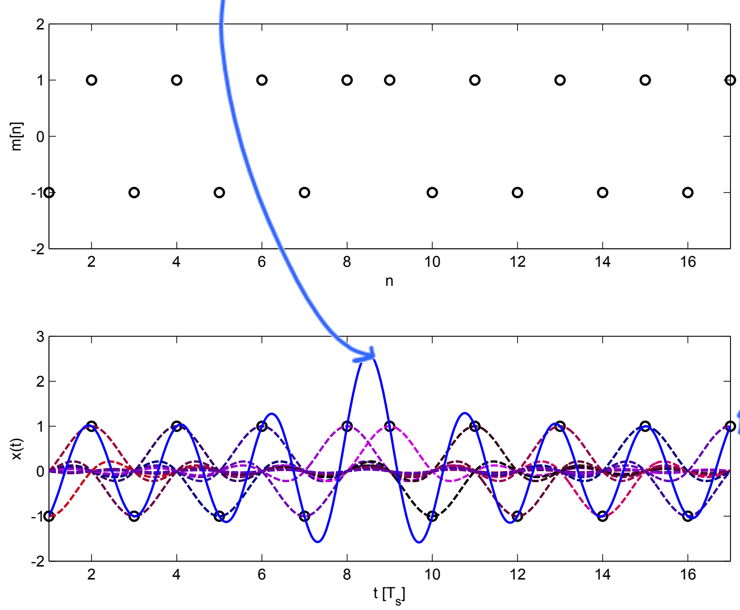


Figure 40: Using the sinc pulse in PAM can cause high peak.

7.9. Because the sinc function may not be a good choice, we now have to consider other pulses that are band-limited and also satisfy (59). To check that a signal is band-limited, we need to look in the frequency domain. However, condition (59) is specified in the time domain. Therefore, we will try to translate condition (59) into a requirement in the frequency domain.

7.10. Note that condition (59) considers $p(t)|_{t=nT}$ which can be thought of as the samples $p[n]$ of the pulse $p(t)$ where the sampling period is $T_s = T$. Recall, from (57), that

$f_s = \frac{1}{T_s}$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_\delta(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s).$$

Therefore,

$\delta(t) \xrightarrow{\mathcal{F}} 1$

$$p_\delta(t) = \overbrace{\sum_{n=-\infty}^{\infty} p[n]\delta(t - nT)}^{\delta(t)} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \overset{1}{\overset{\parallel}{P_\delta(f)}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T}\right). \qquad (60)$$

$$\delta[n] = \begin{cases} 1, & n=0, \\ 0, & n\neq 0. \end{cases}$$

On the LHS, by condition (59), the only nonzero term in the sum is the one with $n = 0$. Therefore, condition (59) is equivalent to $p_\delta(t) = \delta(t)$. However, recall that $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1$. Therefore, we must have $P_\delta(f) \equiv 1$. Hence, to check condition (59), we can equivalently check that the RHS of (60) must be $\equiv 1$.

Note that $P_\delta(f)$ is "periodic" (in the freq. domain) with "period" $\frac{1}{T}$. (Recall that $G_\delta(f)$ is "periodic" (in the freq. domain) with "period" f_s.) Therefore, the checking does not need to be performed across all frequency f. We only need to focus on one period: $|f| \leq \frac{1}{2T}$.

This observation is formally stated as the "Nyquist's criterion" below.

7.11. Nyquist's (first) Criterion for Zero ISI: A pulse $p(t)$ whose Fourier transform $P(f)$ satisfies the criterion

new condition 1 $\longrightarrow$

$$\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T}\right) \equiv T, \qquad |f| \leq \frac{1}{2T} \qquad (61)$$

has sample values satisfying condition (59):

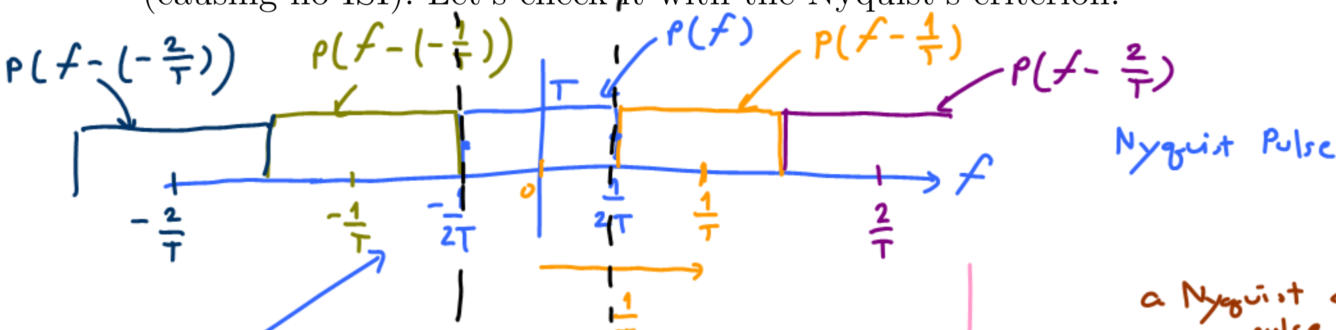
$$p[n] = p(t)|_{t=nT} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

- Using this pulse, there will be no ISI in the sample values of

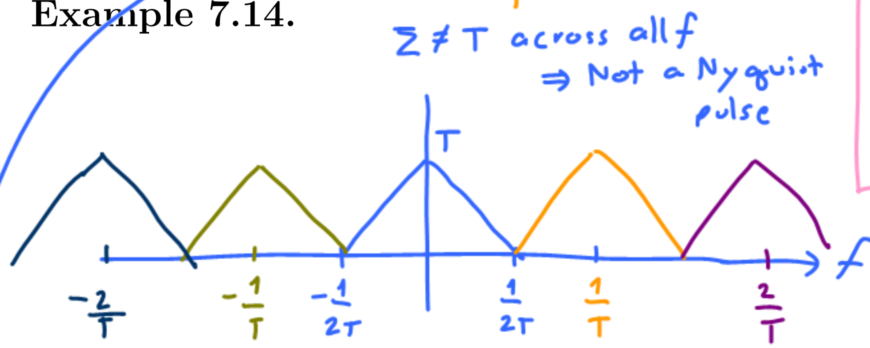
$$y(t) = \sum_{n=-\infty}^{\infty} m[n]\, p(t - nT)$$

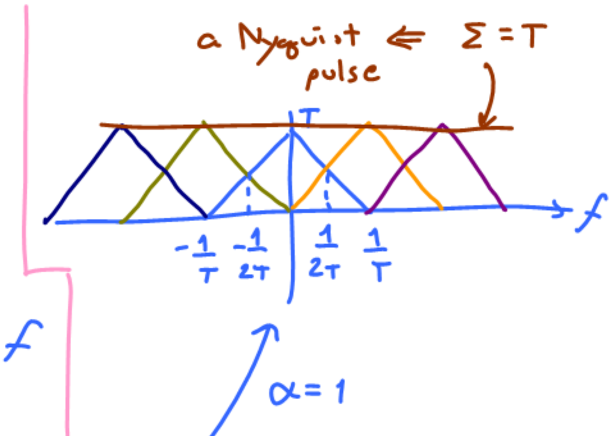
Definition 7.12. A pulse $p(t)$ is a **Nyquist pulse** if its Fourier transform $P(f)$ satisfies (61) above.

Example 7.13. We know that the sinc pulse we used in Example 7.8 works (causing no ISI). Let's check it with the Nyquist's criterion:

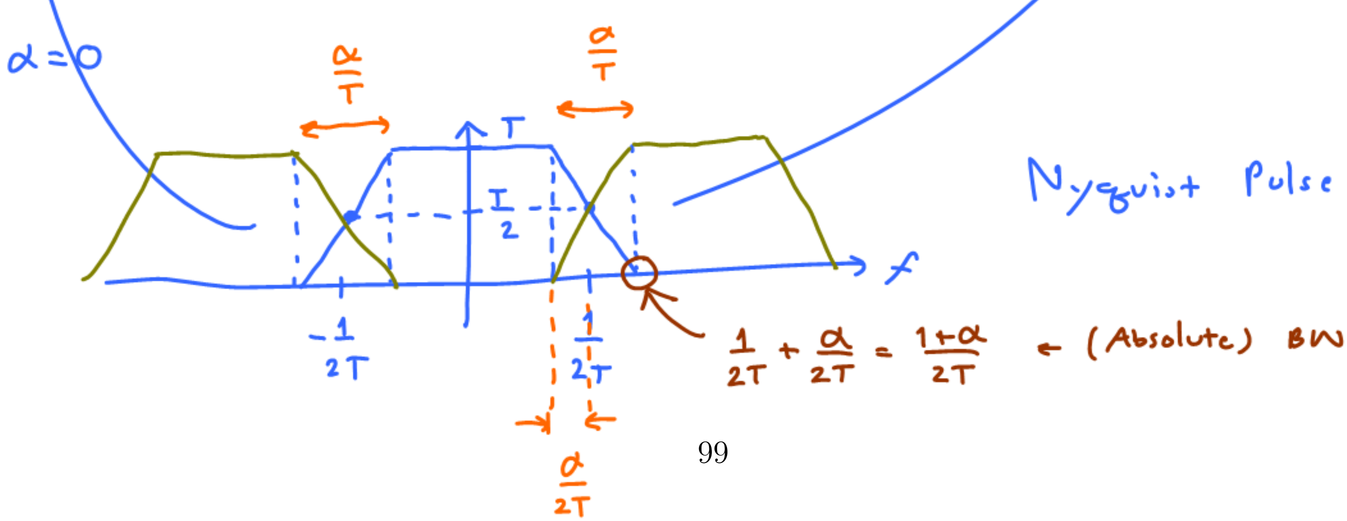


Example 7.14.

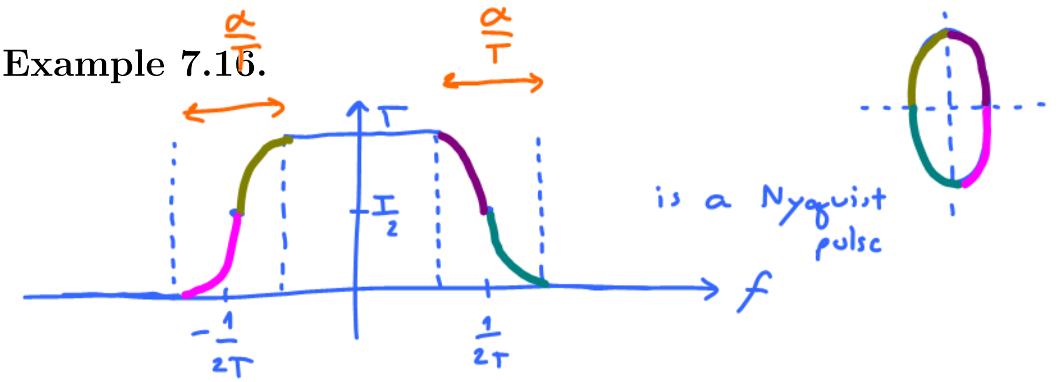




Example 7.15.



Example 7.16.



Example 7.17. An important family of Nyquist pulses is called the **raised cosine** family. Its Fourier transform is given by

$$P_{\mathrm{RC}}(f;\alpha) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left(1+\cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right), & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & |f| \ge \frac{1+\alpha}{2T} \end{cases}$$

with a parameter α called the **roll-off factor**.

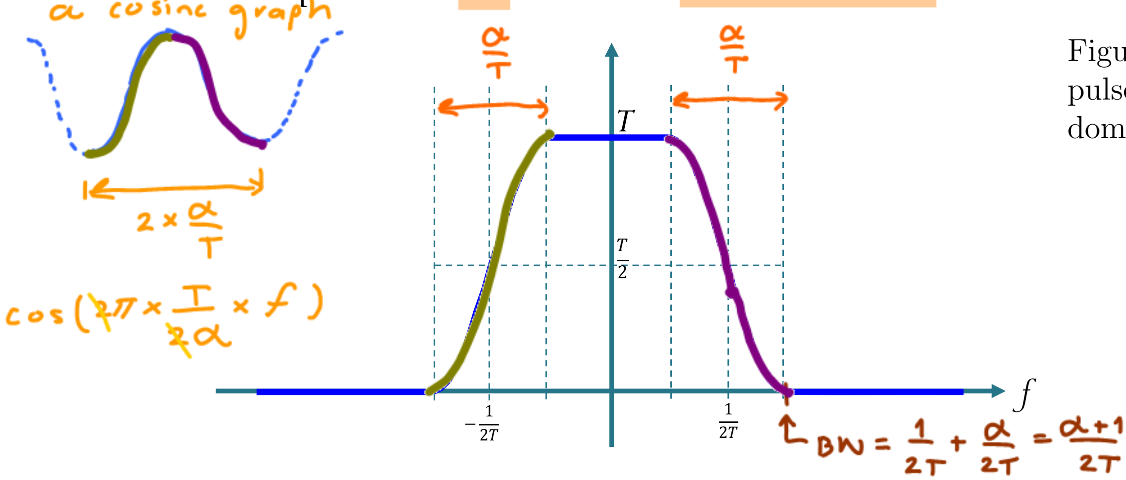


Figure 41: Raised cosine pulse (in the frequency domain)

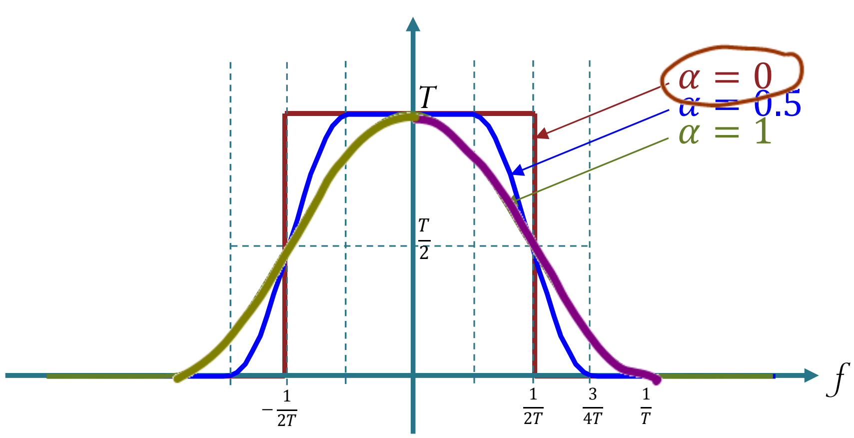


Figure 42: Raised cosine pulse (in the frequency domain) with different values of the roll-off factor

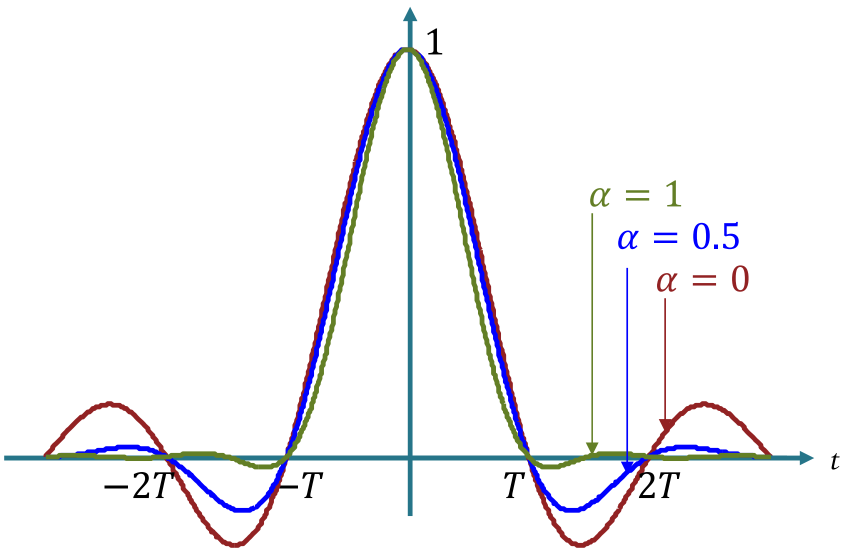


Figure 43: Raised cosine pulse (in the time domain) with different values of the roll-off factor

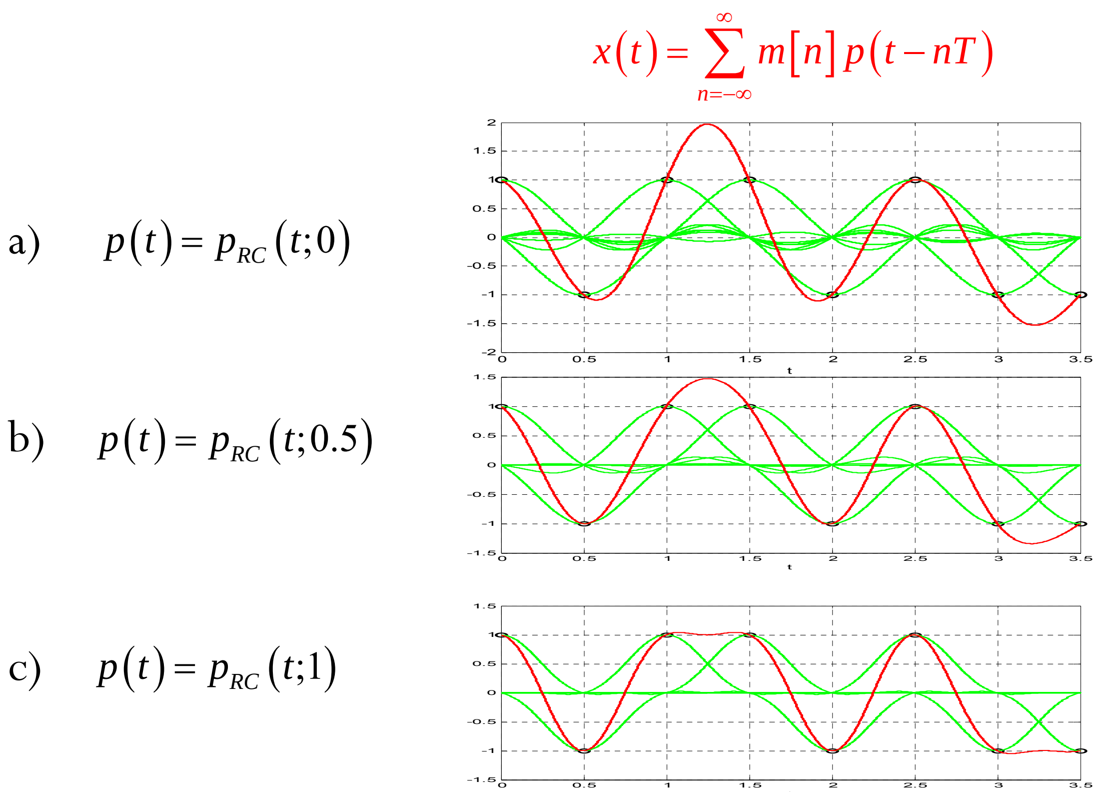


Figure 44: Using the raised cosine pulses in PAM